HOMEWORK THREE

This homework problem set can be accomplished with the help of references. Every problem worths 3 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on December 29 (sharp).

Exercise 1. Recall the following pre-Carleman Similarity Principle (proved in class): if D is a disk around $0 \in \mathbb{C}$ and $u : D \to \mathbb{R}^{2n}$ satisifies

$$\frac{\partial u}{\partial s} + \mathrm{J}(z)\frac{\partial u}{\partial t} + C(z)u(z) = 0$$

for $z \in D$, where $\mathrm{J}(z)$ and $C(z)$ are smoothly parametrized $2n \times 2n$-matrices, then there exist is an open set D' with $0 \in D' \subset D$ and a smooth function $\Phi : D' \to \mathrm{GL}(2n, \mathbb{R})$, $B : D' \to \mathbb{R}^{2n\times 2n}$ and $v : D' \to \mathbb{R}^{2n}$ such that for $z \in D'$, we have

$$v(z) = \Phi(z)u(z) \quad \text{and} \quad \frac{\partial v}{\partial s} + J_0\frac{\partial v}{\partial t} + B(z)v(z) = 0$$

where J_0 is the standard (almost) complex structure in $\mathbb{R}^{2n}$. Prove that this conclusion also holds when $B : D' \to \mathbb{R}^{2n\times 2n}$ is replaced by a bounded (but not necessarily continuous) function $B' : D' \to \{B \in \mathbb{R}^{2n\times 2n} \mid BJ_0 = J_0B\}$.

Exercise 2. Let (X, Ω) be a closed symplectic manifold and $M \subset X$ be a closed hypersurface. Prove that the following two conclusions are equivalent:
(i) The restriction $\omega := \Omega|_M$ admits a stable framing λ (therefore, M admits a stably framed Hamiltonian structure);
(ii) there exist a neighborhood U of M in X and a vector field Y in U transverse to M such that its flow φ_Y^r satisfies $\varphi_Y^r(M)$ is diffeomorphic to M for each $r \in (-\epsilon, \epsilon)$ (for some $\epsilon > 0$) and $(\varphi_Y^r)_*$ preserves the corresponding kernels of the Hamiltonian structures.

Exercise 3. Given a closed symplectic manifold (X, Ω) and a 1-periodic Hamiltonian function $H : S^1 \times X \to \mathbb{R}$. Consider odd-dimensional manifold $M := S^1(t) \times X$. Complete the following problems.
(i) Prove that

$$(\omega, \lambda) := (\Omega + dt \wedge dH, dt)$$

is a stably framed Hamiltonian structure on M (here, notations are defined with appropriate compositions with the pullbacks of projections), and calculate its Reeb vector field.
(ii) For any compatible almost complex structure $J \in \mathcal{J}((\omega, \lambda))$ on the symplectization $\mathbb{R} \times M$ (which can be identified with a smoothly S^1-parametrized family

of compatible almost complex structures $\{J_t\}_{t\in S^1}$ on (X,ω), due to the translation invariant property), consider a J-holomorphic cylinder

$$u : (\mathbb{R}\times S^1, j) \to (\mathbb{R}\times M, J)(= ((\mathbb{R}\times S^1)\times X, J)).$$

Write $u = (\varphi, \tilde{v})$ where $\varphi : \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$ and $\tilde{v} : \mathbb{R}\times S^1 \to X$. Assume that φ is injective, so after a reparametrization one can write $u = (\mathbb{1}, v)$. Write out the express of the partial differential equation that v should satisfy and justify your answer with details.

Exercise 4. Given a closed symplectic manifold (M,ω) and a compatible almost complex structure J, there exists a constant $\hbar > 0$ (only depending on M, ω, and J) such that for any $\epsilon > 0$, there is a constant $\eta > 0$ such that for any J-holomorphic curve $u : (D(1), j_{\rm std}) \to (M, J)$ with $E(u) = \mathrm{Area}(u) < \hbar$, we have $u(D(\eta)) \subset B(u(0), \epsilon)$. Here, $D(\eta)$ denotes the standard disk in $\mathbb{C}$ centered at the origin with radius η, and $B(u(0), \epsilon)$ denotes the ball in M centered at $u(0)$ with radius ϵ (with respect to the metric $\omega(\cdot, J\cdot)$).

Exercise 5. Let $(M^{2n-2\geq 2}, \omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$ and σ be an area form on S^2. Prove that if there exists a symplectic embedding

$$\iota : (B^{2n}(0,r), \omega_{\rm std}) \to (S^2\times M, \sigma\oplus\omega),$$

then $\int_{S^2}\sigma \geq \pi r^2$.

Note that to solve this problem, one can/should use the following result for free on the existence of a pseudo-holomorphic curve: in the setting above, there exits an almost complex structure J on $(S^2\times M, \sigma\oplus\omega)$ with $\iota^*J = \sqrt{-1}$ on $B^{2n}(0,r)$ and a J-holomorphic sphere $u : (S^2, j_{\rm std}) \to (S^2\times M, J)$ with $[\mathrm{im}(u)] = [S^2\times\{\mathrm{pt}\}] \in H_2(S^2\times M)$ whose image contains $\iota(0)$.