Ex 1:

For $\tilde{A} \in M_{\mathbb{R}}(k,n)$. WLOG. $[\tilde{A}] = [A]$ in $Gr_{\mathbb{R}}(k,n)$.

$\exists\ \mathcal{U}_I$. φ_I. $[\tilde{A}] \in \mathcal{U}_I$

$$A = \begin{pmatrix} 1 & & & a_{1k+1} & \cdots & a_{1n} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & a_{kk+1} & \cdots & a_{kn} \end{pmatrix}^t$$

$$\& \quad \varphi_I([\tilde{A}]) = \varphi_I([A]) = \begin{pmatrix} a_{1k+1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{kk+1} & \cdots & a_{kn} \end{pmatrix}^t$$

$$= (a_{1,k+1}, \cdots a_{1n}, \cdots a_{k,k+1}, \cdots a_{kn}). \in \mathbb{R}^{k(n-k)}.$$

For $\pi : (V, v) \mapsto V$. open cover $\{\mathcal{U}_I\}_I$ of $Gr_{\mathbb{R}}(k,n)$ define

$$\Phi_I : \pi^{-1}(\mathcal{U}_I) \longrightarrow \mathcal{U}_I \times \mathbb{R}^k$$
$$([A], v) \mapsto ([A], (v^1, \dots v^k)).$$

where v^i satisfies

$$v = P_I^{-1} \begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}, \quad P_I \in GL(n, \mathbb{R}).$$

$\pi^{-1}([A]) = \{v \in \mathbb{R}^n : v \in [A]\}$. is a k vector space &

$$\Phi_I\big|_{\pi^{-1}([A])} : \pi^{-1}([A]) \longrightarrow \{[A]\} \times \mathbb{R}^k$$

is a linear isomorphism.

Now for $\mathcal{U}_I \cap \mathcal{U}_J \neq \emptyset$.

$$\Phi_J \circ \Phi_I^{-1}([A], (v^1, \dots v^k)) = ([A], (w^1, \dots w^k)).$$

Recall 李's solution:

For

$$P_J A = P_J P_I^{-1}(P_I A) = P_J P_I^{-1}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}$$

$$= \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}$$

$$\begin{pmatrix} Id \\ \varphi_J([A]) \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}\left(R_1 + R_2\varphi_I([A])\right)^{-1}$$

$$P_J^{-1}\begin{pmatrix} Id \\ \varphi_J([A]) \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = P_J^{-1} \cdot P_J \cdot P_I^{-1}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}\left(R_1+R_2\varphi_I([A])\right)^{-1}\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$$

$$= P_I^{-1}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}\left(R_1+R_2\varphi_I([A])\right)^{-1}\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$$

$$= P_I^{-1}\begin{pmatrix} Id \\ \varphi_I([A]) \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

$\Rightarrow g_{JI}([A]) = \left(R_1 + R_2\varphi_I([A])\right)^{-1} : \mathbb{R}^k \to \mathbb{R}^k$

$g_{JI} : U_I \cap U_J \to GL(k, \mathbb{R})$. is smooth.

Ex 2:

LHS: $D_{[X,Y]} f = D_X D_Y f - D_Y D_X f. = ① - ②.$

Take $X = X_i E^i$. $Y = Y_i E^i$. $g_i = D_X Y_i - D_Y X_i$.

Need to show:

$① - ② = D_{g_i E^i}(f) = g_i E^i(f)$. $\forall f \in \Gamma(TM)$.

$① = D_X(D_{Y_iE^i} f) = D_{X_jE^j}(Y^i E_i(f))$

$= X_j E^j(Y^i) E_i(f) + X_j Y_i E^j E^i(f).$

$② = D_Y D_X f = Y_j E^j(X_i) E^i(f) + Y_j X_i E^j E^i(f).$

$D_{[X,Y]} f = ① - ② = X_j E^j(Y^i) E_i(f) - Y_j E^j(X_i) E^i(f)$

$= g_i E^i(f) = D_{g_iE^i}(f).$

since $g_i = D_X Y_i - D_Y X_i = X_j E^j(Y_i) - Y_j E^j(X_i)$.

The proof is done.

For $X = (-y, x, 0)$ $Y = (0, -z, y)$.

$g_1 = 0 - (-z)\times(-1) = -z$

$g_2 = 0 - 0 = 0.$

$g_3 = x - 0 = x.$

$\Rightarrow [X,Y](x,y,z) = (-z, 0, x).$

Ex 3:

(1). $T^2 = S^1 \times S^1$. For $(\theta, \varphi) \in T^2$. $\theta, \varphi \in [0, 2\pi)$.

$X_1 = \frac{\partial}{\partial\theta}$, $X_2 = \frac{\partial}{\partial\varphi}$. do not have any zero's.

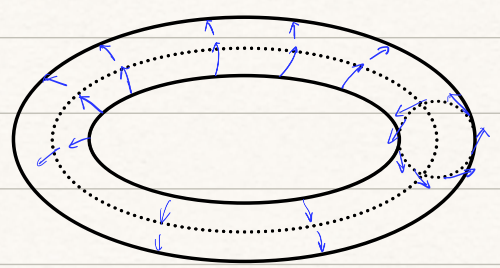
Explanation:

For torus in $\mathbb{R}^3$.

$$\begin{cases} x = (R + r\cos\theta)\cos\varphi \\ y = (R + r\sin\theta)\sin\varphi \\ z = r\sin\theta \end{cases}$$

Then $\frac{\partial}{\partial\theta}$ is

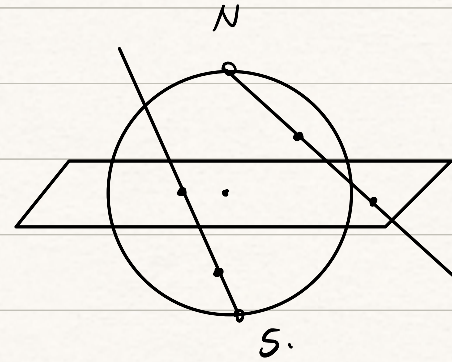
$(-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, r\cos\theta,).$



12). Consider stereographic projection:

$\sigma_1 : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ $\quad N = (0.0.1)$.

$\sigma_1(x.y.z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.



$\sigma_1^{-1}(u.v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}. \frac{u^2+v^2-1}{u^2+v^2+1}\right)$.

$\sigma_2 : S^2 \setminus \{S\} \to \mathbb{R}^2$. $\quad S = (0.0.-1)$.

$\sigma_2(x.y.z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$.

$\sigma_2^{-1}(u.v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}. \frac{1-u^2-v^2}{u^2+v^2+1}\right)$.

$\sigma_1^{-1} : \mathbb{R}^2 \to S^2 \setminus N$. $\quad S^2\setminus N \cap S^2 \setminus S = S^2 \setminus \{N.S\}$.

$\quad (u.v) \longmapsto (x.y.z)$.

$d(\sigma_1^{-1}) : \Gamma(T\mathbb{R}^2) \to \Gamma(T(S^2\setminus N))$

$\quad \frac{\partial}{\partial u} \longmapsto d(\sigma_1^{-1})(\frac{\partial}{\partial u})$.

Consider $\quad \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$.

$\sigma_2 \circ \sigma_1^{-1} : \quad (u.v) \mapsto (u_1.v_1) = \left(\frac{u}{u^2+v^2}. \frac{v}{u^2+v^2}\right)$.

$\sigma_1 \circ \sigma_2^{-1} : \quad (u_1.v_1) \mapsto (u.v) = \left(\frac{u_1}{u_1^2+v_1^2}, \frac{v_1}{u_1^2+v_1^2}\right)$.

For $p \in \mathbb{R}^2\setminus\{0\}$.

$$d(\sigma_2\circ\sigma_1^{-1})(\frac{\partial}{\partial u})(p) = \frac{\partial u_1}{\partial u}\frac{\partial}{\partial u_1} + \frac{\partial v_1}{\partial u}\frac{\partial}{\partial v_1}$$

$$= \frac{v^2-u^2}{(u^2+v^2)^2}\frac{\partial}{\partial u_1} - \frac{2uv}{(u^2+v^2)^2}\frac{\partial}{\partial v_1} \qquad (*).$$

$$= (v_1^2-u_1^2)\frac{\partial}{\partial u_1} - 2u_1v_1\frac{\partial}{\partial v_1}$$

$$d(\sigma_2\circ\sigma_1^{-1})(\frac{\partial}{\partial v})(p) = \frac{\partial u_1}{\partial v}\frac{\partial}{\partial u_1} + \frac{\partial v_1}{\partial v}\frac{\partial}{\partial v_1}.$$

$$= \frac{-2uv}{(u^2+v^2)^2}\frac{\partial}{\partial u_1} + \frac{u^2-v^2}{(u^2+v^2)^2}\frac{\partial}{\partial v_1}$$

$$= -2u_1v_1\frac{\partial}{\partial u_1} + (u_1^2-v_1^2)\frac{\partial}{\partial v_1}.$$

Then define

$$X_q = \begin{cases} d(\sigma_1^{-1})(\sigma_1(q))\left(\frac{\partial}{\partial u}\right). & q \in S^2 \setminus \{N\}. \\ d(\sigma_2^{-1})(\sigma_2(q))\left((v_1^2 - u_1^2)\frac{\partial}{\partial u_1} - 2u_1 v_1 \frac{\partial}{\partial v_1}\right) & q \in S^2 \setminus \{S\}. \end{cases}$$

By $(*)$. X is well defined on $S^2 \setminus \{N\} \cap S^2 \setminus \{S\}$.
& is smooth on S^2. (Smoothness from local coordinate).
Now $\frac{\partial}{\partial u} \neq 0. \Rightarrow X_q \neq 0.\ \forall q \in S^2 \setminus \{N\}$.
For $q = N$. $u_1 = v_1 = 0$. from the calculation
$X_q = 0$.
So $X \in \Gamma(TS^2)$ with only one zero.

Ex 4.

For $S^3 = \{(x.y.z.w):\ x^2+y^2+z^2+w^2 = 1\}$.

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}.$$
$$X_2 = -z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial w}$$
$$X_3 = -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w}.$$

Check $X_i \in \Gamma(T(S^3))$ note $\vec{n} = (x.y.z.w)$ is normal vector:

$$\begin{cases} -yx + xy - wz + zw = 0. \\ -zx + wy + xz - yw = 0. \\ -wx - zy + yz + xw = 0. \end{cases}$$

smoothness is obvious.

$\forall p \in S^3$. $p = (x.y.z.w)$. if $\exists$ a.b.c. $a^2+b^2+c^2 \neq 0$ s.t.
$aX_1(p) + bX_2(p) + cX_3(p) = 0$.
Take $Y = aX_1(p) + bX_2(p) + cX_3(p)$.
$\langle Y.Y\rangle = a^2\langle X_1.X_1\rangle + b^2\langle X_2.X_2\rangle + c^2\langle X_3.X_3\rangle + 0$
($X_1.X_2.X_3$ are pairwise orthogonal at each point).
$\Rightarrow a^2+b^2+c^2 = 0$ ↯

$\Rightarrow \forall p \in S^3$. $\{X_1(p), X_2(p), X_3(p)\}$ linearly independent.
$\dim(T_p S^3) = 3$. $\Rightarrow$ forms a basis at $T_p(S^3)$.

Ex 5:

$\forall w \in W$. consider bilinear map

$$f_z : \mathcal{U} \times V \longrightarrow \mathcal{U} \otimes (V \otimes W).$$
$$(u, v) \longmapsto u \otimes (v \otimes w).$$

By the universal property, $\exists!$ homomorphism

$$\varphi_z : \mathcal{U} \otimes V \to \mathcal{U} \otimes (V \otimes W). \quad \text{s.t.}$$

$$\begin{tikzcd} \mathcal{U} \times V \arrow[d] \arrow[dr, "f_z"] & \\ \mathcal{U} \otimes V \arrow[r, dotted, "\varphi_z."'] & \mathcal{U} \otimes (V \otimes W) \end{tikzcd}$$

Then consider

$$\tilde{\varphi} : (\mathcal{U} \otimes V) \times W \longrightarrow \mathcal{U} \otimes (V \otimes W).$$
$$(x, w) \longmapsto \varphi_w(x).$$

$\tilde{h}$ is bilinear map & by the universal property again:

$$\varphi : (\mathcal{U} \otimes V) \otimes W \to \mathcal{U} \otimes (V \otimes W)$$
$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

Similarly, construct homomorphism

$$\psi : \mathcal{U} \otimes (V \otimes W) \longrightarrow (\mathcal{U} \otimes V) \otimes W.$$
$$u \otimes (v \otimes w) \longmapsto (u \otimes v) \otimes w$$

$\varphi \circ \psi = Id_{\mathcal{U} \otimes (V \otimes W)}$. $\quad \psi \circ \varphi = Id_{(\mathcal{U} \otimes V) \otimes W}$

So φ, ψ are isomorphisms satisfying the condition.

Ex 6:

$\Leftarrow$: If (a_{ij}) has rank 1. $\exists B \in GL(n)$ s.t.

$$AB^{-1} = \begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & & & \\ \vdots & & & \\ c_{n1} & 0 & \cdots & 0 \end{pmatrix}_{n\times m}.$$

$$a_{ij} = b_{ik}c_{kj} = \begin{cases} b_{ik}c_{k1} & j=1 \\ 0 & j\neq 1 \end{cases}$$

$\Rightarrow x = \sum b_{ik}c_{k1}\, e_i \otimes f_1 = \left(\sum b_{ik}c_{k1} e_i\right) \otimes f_1$

x is decomposable.

$\Rightarrow$: If x is decomposable, set

$$x = \left(\sum_{i=1}^{n} x_i e_i\right) \otimes \left(\sum_{j=1}^{m} y_j f_j\right).$$

$$= \sum_{i,j} x_i y_j\, e_i \otimes f_j.$$

Note $\{e_i \otimes f_j\}$ is a basis of $V \otimes W$.

$$a_{ij} = x_i y_j.$$

$$(a_{ij}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (y_1 \cdots y_m).$$

$0 < \operatorname{rank}(a_{ij}) \leq \operatorname{rank}((y_1 \cdots y_m)) = 1.$ (Non trivial x).

$\Rightarrow \operatorname{rank}(a_{ij}) = 1.$

Ex 7.

$A \otimes B = (A \otimes Id_{l\times l})(Id_{k\times k} \otimes B) \in GL(kl, \mathbb{R}).$

$$\det(A \otimes Id_{l\times l}) = \det(Id_{l\times l} \otimes A) = \det \underbrace{\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}}_{l\text{块}}$$

$$= (\det(A))^l$$

$\det(Id_{k\times k} \otimes B) = (\det(B))^k$

$\Rightarrow \det(A \otimes B) = (\det(A))^l (\det(B))^k.$

Ex 8.

$\forall X \in \Gamma(TM)$. $\forall x \in M$. $X(x)$. $JX(x)$ are always linear independent since $J_x^2 = -Id$.

$N_J(X.JX) = \underbrace{[X.JX] + J[JX.JX] + J[X.-X] - [JX.-X]}_{=0}$.

$N_J(X.X) = [X.X] + J[JX.X] + J[X.JX] - [JX.JX] = 0$.

$\dim(\Sigma) = \dim(M) = 2$. $\Rightarrow \forall x$. $\dim T_xM = 2$.

$\Rightarrow N_J \equiv 0$. J integrable.

Ex 9: Levi-Civita connection

Prove by direct calculation: write $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$ for short.

$\forall X. Y. Z \in \Gamma(TM)$.

$g(\nabla_X Y. Z) = \langle \nabla_X Y. Z\rangle$

$\overset{①}{=} X(\langle Y. Z\rangle) - \langle Y. \nabla_X Z\rangle$

$\overset{②}{=} X(\langle Y. Z\rangle) - \langle Y. \nabla_Z X + [X.Z]\rangle$.

$= X(\langle Y. Z\rangle) - \langle Y. \nabla_Z X\rangle - \langle Y, [X.Z]\rangle$.

$\overset{①}{=} X(\langle Y. Z\rangle) - Z\langle Y. X\rangle + \langle \nabla_Z Y. X\rangle - \langle Y, [X.Z]\rangle$.

$\overset{②}{=} X(\langle Y. Z\rangle) - Z\langle Y. X\rangle + \langle [Z.Y]. X\rangle + \langle \nabla_Y Z, X\rangle - \langle Y, [X.Z]\rangle$.

$\overset{①}{=} X(\langle Y. Z\rangle) - Z\langle Y. X\rangle + \langle [Z.Y]. X\rangle + Y\langle Z.X\rangle - \langle Z. \nabla_Y X\rangle - \langle Y, [X.Z]\rangle$.

$\overset{②}{=} X(\langle Y. Z\rangle) - Z\langle Y. X\rangle + \langle [Z.Y]. X\rangle + Y\langle Z.X\rangle - \langle Z. \nabla_X Y\rangle - \langle Z, [Y.X]\rangle - \langle Y, [X.Z]\rangle$.

So we have proved:

$$2\langle \nabla_X Y. Z\rangle = X(\langle Y. Z\rangle) - Z\langle Y. X\rangle + Y\langle Z.X\rangle + \langle [Z.Y]. X\rangle - \langle Z, [Y.X]\rangle - \langle Y, [X.Z]\rangle.$$

$$= X(\langle Y. Z\rangle) + Y(\langle Z. X\rangle) - Z\langle X. Y\rangle. - \langle X. [Y.Z]\rangle + \langle Y, [Z.X]\rangle + \langle Z, [X.Y]\rangle.$$

The connection is decided uniquely by $g(\cdot,\cdot)$.

& satisfies (i). & (ii).

Ex 10:

(1). In local coordinate $\nabla F := g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j}$

$$\Rightarrow g(\nabla F, X) = g^{ij} \frac{\partial F}{\partial x^i} X^k g_{jk} = \frac{\partial F}{\partial x^i} X^k \delta^i_k.$$

$$= X^i \frac{\partial F}{\partial x^i} = D_{X^i \frac{\partial}{\partial x^i}} F = D_X F.$$

So ∇F satisfies the condition.

If $Y = Y^i \frac{\partial}{\partial x^i}$ s.t. $g(Y, X) = D_X F$.

$$g_{ij} Y^i X^j = X^i \frac{\partial F}{\partial x^i} \quad \forall X \in \Gamma(TM) \Rightarrow g_{ij} Y^i = \frac{\partial F}{\partial x^j}$$

$$g_{ij} Y^i g^{jk} = \frac{\partial F}{\partial x^j} g^{jk} \Rightarrow Y^i = g^{ij} \frac{\partial F}{\partial x^j} \Rightarrow Y = g^{ij} \frac{\partial F}{\partial x^j} \frac{\partial}{\partial x^i}$$

which gives uniquenss.

(2). Consider the directional derivative:

$$D_{\nabla F} F = g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j}(F) = g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = |\nabla F|_g^2 \geq 0,$$

g_{ij} is positive - definite $\Rightarrow$ g^{ij} is positive-definite.

$\Rightarrow D_{\nabla F} F \geq 0$. F is non-decreasing along ∇F.

(3). Only needs to calculate $(g_{r\theta})$ & $(g^{r\theta})$.

$x = r\cos\theta$. $y = r\sin\theta$.

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos\theta\, \partial_x + \sin\theta\, \partial_y$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = -r\sin\theta\, \partial_x + r\cos\theta\, \partial_y$$

$$g_{rr} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \qquad g_{r\theta} = g_{\theta r} = -r\cos\theta\sin\theta + r\cos\theta\sin\theta = 0.$$

$$g_{\theta\theta} = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \quad \Rightarrow \quad g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$g^{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \qquad \nabla F = \frac{\partial F}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial F}{\partial \theta}\frac{\partial}{\partial \theta}$$