# HOMEWORK ONE

**This homework problem set can be accomplished with the help of references. Every problem worths 2 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on October 30 (sharp).**

**Exercise 1**. Let $\mathbb{D}^* = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$, endowed with the standard symplectic structure $dx \wedge dy$ where $z = x + \sqrt{-1}y$. Give an explicit formula for a symplectomorphism $\varphi : \mathbb{D}^* \to \mathbb{D}^*$ that turns $\mathbb{D}^*$ "inside out" in the sense that if $\{z_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{D}^*$ approaching to $0 \in \mathbb{C}$, then $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Please justify that your $\varphi$ is indeed a symplectomorphism. (Hint: it will be easier to work with polar coordinate first.)

**Exercise 2**. Let $(M, \omega)$ be a symplectic manifold with an $\omega$-compatible $J$ and $(\Sigma, j, \mathrm{dvol}_\Sigma)$ be a closed Riemannian surface with a fixed volume form $\mathrm{dvol}_\Sigma$. The energy of a smooth map $u : \Sigma \to M$ is defined as follows:

$$E(u) := \frac{1}{2} \int_\Sigma |\mathrm{d}u|_J^2 \,\mathrm{dvol}_\Sigma$$

where $|\cdot|_J$ is the norm under the metric $\omega(\cdot, J\cdot)$. Prove that

$$E(u) = \int_\Sigma |\bar{\partial}_J(u)|_J^2 \,\mathrm{dvol}_\Sigma + \int_\Sigma u^*\omega.$$

In particular, if $u$ is $J$-holomorphic, then $E(u) = \int_\Sigma u^*\omega$.

**Exercise 3**. Let $(X, \xi = \ker\alpha)$ be a contact manifold with a fixed contact 1-form $\alpha$. Consider the following functional on the loop space of $X$:

$$\gamma \in C^\infty(S^1, X) \mapsto \mathcal{A}_\alpha(\gamma) := \int_\gamma \alpha.$$

Complete the following question:

(1) Calculate the critical points of $\mathcal{A}_\alpha$ and identify them with well-known objects in contact geometry.
(2) Calculate the Hessian of $\mathcal{A}_\alpha$ and determine when a critical point is non-degenerate (in the Morse sense).

**Exercise 4**. For any almost complex manifold $(M, J)$, prove the following two conclusions:

(1) There exists a $J$-compatible Riemannian metric $g$ in the sense that for any $X, Y \in \Gamma(M, TM)$, we have $g(JX, JY) = g(X, Y)$.

(2) Take the Levi-Civita connection $\nabla$ of the metric $g$ in (1) and consider the following affine connection on the tangent bundle

$$\widetilde{\nabla} Y := \nabla Y - \frac{1}{2} J(\nabla J)(Y).$$

Here, $\nabla J$ means the induced connection (still denoted by $J$) on the bundle $\mathrm{End}(TM) \to M$ acting on the section $J$. Prove that the induced connection from $\widetilde{\nabla}$ on $T^*M \otimes T^*M \to M$, still denoted by $\widetilde{\nabla}$ satisfies $\widetilde{\nabla} g = 0$.

**Exercise 5**. We will prove a simple version of the Arnold conjecture (on the number of fixed points of a Hamiltonian diffeomorphism) via the following three steps.

(1) Let $x(t) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$ be a smooth map with mean value zero. Then we have the following $L^2$-estimate:

$$\|x\|_{L^2} \leq \frac{1}{2\pi} \|\dot{x}\|_{L^2}.$$

(Hint: use Fourier expansion.)

(2) Use (1) to prove that given a compactly supported function $H : \mathbb{R}^{2n} \to \mathbb{R}$ as an autonomous Hamiltonian function on $\mathbb{R}^{2n}$ with respect to the standard symplectic structure $\omega_{\mathrm{std}}$, if its Hessian is sufficiently small, then the only solutions of the 1-periodic orbit of the Hamiltonian flow of the corresponding Hamiltonian vector field $X_H$ are the constant ones.

(3) Use (2) prove that for any $C^2$-small autonomous Hamiltonian function $H$ and also Morse[1] on a symplectic manifold $(M^{2n}, \omega)$, the Arnold conjecture holds:

$$\#\mathrm{Fix}(\phi_H^1) \geq \sum_{i=1}^{2n} b_i(M; \mathbb{Z}_2).$$

(You are free to use the Darboux theorem in symplectic geometry: locally any symplectic manifold can be identified with the standard Euclidean space.)

[1]Strictly speaking, the original hypothesis for this version of the Arnold conjecture is that such Hamiltonian function is non-degenerate (in some sense, not explicitly elaborated in class). In fact, one can verify that, under the condition that all 1-periodic orbits of $H$ are the constant ones, the non-degeneracy of this $H$ is equivalent to $H$ being Morse.