MATH5003P FINAL EXAM, FALL 2024

This is a CLOSED-BOOK exam, and electronic devices and collaborations are NOT allowed. There are 10 problems, 10 points for each, 100 points in total. Please provide enough details to justify your answers. The exam time is January 6, 2025, from 19:30 to 22:00.

NAME_______________ **STUDENT ID.** _______________

Problem 1. Complete the following three problems on manifold:
(a) [3 points] Write down the definition of a manifold.
(b) [3 points] Prove that $S^3 \times \mathbb{R}P^2$ is a manifold. Provide necessary details.
(c) [4 points] Determine whether $S^3 \times \mathbb{R}P^2$ is orientable or not. Justify your answer.

Problem 2. Answer only YES or NO for the following five statements. There is **no need** to justify your answers.
(a) [2 points] Any Lie group is orientable.
(b) [2 points] Two Lie groups are isomorphic to each other if and only if their Lie algebras are isomorphic to each other.
(c) [2 points] There exists a smooth embedding from S^3 to $\mathbb{R}^3$.
(d) [2 points] For any smooth map $F : M \to \mathbb{R}$, there always exists a non-empty interval $I \subset \mathbb{R}$ that contains no critical values of F at all.
(e) [2 points] The tangent bundle of the 6-dimensional sphere S^6 is trivial, i.e., there exists a homeomorphism $\Phi : TS^6 \simeq S^6 \times \mathbb{R}^6$ such that restrictions to the fibers are linear isomorphisms.

Problem 3. Complete the following four questions on stating theorems.
(a) [2 points] State the Constant Rank Theorem.
(b) [2 points] State Stokes' theorem for a manifold with boundary.
(c) [3 points] State Ado-Iwasawa's theorem on Lie algebra representation.
(d) [3 points] State Hodge theorem on harmonic forms (and decomposition result).

Problem 4. Complete the following questions about Mayer–Vietoris sequence.
(a) [2 points] Suppose $M = U \cup V$, then write down the long exact sequence of the Mayer–Vietoris sequence (of de Rham cohomology groups).

(b) [4 points] Recall that the Euler characteristic of M is defined by $\chi(M) := \sum_{k=0}^{\dim M}(-1)^k \dim_{\mathbb{R}} H^k_{\mathrm{dR}}(M;\mathbb{R})$. Prove that if $M = U \cup V$, then

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V).$$

(c) [4 points] Prove that $\chi(M \times S^2) = 2\chi(M)$. (Hint: $M \times S^2 = U \cup V$ where both U and V are diffeomorphic to $M \times \mathbb{R}^2$.)

Problem 5. Complete the following three questions about degree.
(a) [2 points] Let M, N be two orientable manifolds of dimension n, and $F : M \to N$ be a proper smooth map. Write down the definition of $\deg(F)$.
(b) [4 points] On $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ where $\mathbb{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, for map $F : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $F(z, w) = (w, \overline{z})$ where $\overline{z}$ denotes the complex conjugate of z in $\mathbb{C}$, compute its degree $\deg(F)$.
(c) [4 points] Suppose $\mathbb{S}^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \,|\, \sum_i x_i^2 = 1\}$. If $F : \mathbb{S}^n \to \mathbb{S}^n$ does *not* have a fixed point, the prove $\deg(F) = (-1)^{n+1}$. (Hint: consider homotopy $(1-t)F(x) + t(-x)$ for $x \in \mathbb{S}^n$ and $t \in [0, 1]$.)

Problem 6. Complete the following questions about exponential map.
(a) [2 points] Let G be a Lie group with its Lie group denoted by $\mathfrak{g}$. Write down the definition of the exponential map $\exp : \mathfrak{g} \to G$.
(b) [4 points] Prove that for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, the exponential map satisfies $\exp(X)^n = \exp(nX)$. Please provide necessary details.
(c) [4 points] Under the notation in (a), for any given $X, Y \in \mathfrak{g}$, we have

$$\lim_{n\to\infty} \left(\left(\exp\left(\frac{t}{n}X\right)\right) \left(\exp\left(\frac{t}{n}Y\right)\right) \right)^n = \exp(t(X+Y))$$

when t is sufficiently small.

Problem 7. Construct examples for the following four questions. Please justify your answer, explaining the "but *not*..." part.
(a) [2.5 points] $F : \mathbb{R} \to \mathbb{R}^2$ is injective but *not* immersion.
(b) [2.5 points] $F : \mathbb{R} \to \mathbb{R}$ is surjective but *not* submersion.
(c) [2.5 points] $F : \mathbb{R} \to \mathbb{R}^2$ is immersion but *not* injective.
(d) [2.5 points] $F : \mathbb{R} \to \mathbb{R}$ is submersion but *not* surjective.

Problem 8. Complete the following questions about Morse function.
(a) [3 points] For a smooth map $F : M^n \to \mathbb{R}$, its critical point $x \in M$ is called *non-degenerate* if locally near x, in coordinate $(x_1, ..., x_n)$ the Hessian

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)(x) \text{ is non-degenerate (i.e., invertible).}$$

For the following two functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = x^2 + y^2 \quad \text{and} \quad G(x, y) = x^2y^2$$

where $(0, 0) \in \mathbb{R}^2$ is a critical point for both F and G. Pick from F and G the one where point $(0, 0)$ is non-degenerate.

(b) [3 points] Prove that any non-degenerate critical point is isolated, i.e., it admits a neighborhood that contains no other critical point.

(c) [4 points] A *Morse function* is a function where all of its critical points are non-degenerate. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth map. Then for almost every vector $a = (a_1, ..., a_n) \in \mathbb{R}^n$, the function

$$F_a(x) := F(x) - a_1x_1 - \cdots a_nx_n$$

is a Morse function. Here "almost" means that those which do *not* make F_a be Morse functions form a Lebesgue measure zero set in $\mathbb{R}^n$. (Hint: Consider the following map $\tau(F) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\tau(F)(x) = \left(\frac{\partial F}{\partial x_1}(x), ..., \frac{\partial F}{\partial x_n}(x)\right),$$

then apply Morse-Sard's Theorem.)

Problem 9. Let V be a $2n$-dimensional vector space and $\omega \in \bigwedge^2 V^*$. Recall that ω is non-degenerate if for any non-zero $v \in V$, there always exists $w \in V$ such that $\omega(v, w) \neq 0$. Go through the following steps to prove a "strong Lefschetz" type result for ω: for any $0 \leq k \leq n$, the map by wedging with $\omega \wedge \cdots \wedge \omega$ (for k-times),

$$\wedge\omega^k : \bigwedge^{n-k} V^* \to \bigwedge^{n+k} V^* \quad \alpha \mapsto \alpha \wedge \omega^k \tag{1}$$

is an isomorphism. The following argument is due to E. Calabi.

(a) [3 points] Prove that ω is non-degenerate if and only if $\omega \wedge \cdots \wedge \omega$ (wedging n-times) is a non-zero element in $\bigwedge^{2n} V^*$.

(b) [2 points] Prove that it suffices to show that the map in (1) above is injective.

(c) [5 points] Carry out the rest of the proof by induction. Start from $k = n$ and then assume for $k \in \{1, ..., n-1\}$ the injectivity of map in (1) holds, explicitly, for $\alpha \in \bigwedge^{n-k} V^*$, we have that $\alpha \wedge \omega^k = 0$ implies $\alpha = 0$. Prove that this also holds for $k - 1$, that is,

$$\alpha \wedge \omega^{k-1} = 0 \text{ for } \alpha \in \bigwedge^{n-k+1} V^* \implies \alpha = 0.$$

Then by induction (from $k = n$ back to $k = 1$), we obtain the conclusion. (Hint: $\alpha \wedge \omega^{k-1} = 0$ implies that $\alpha \wedge \omega^k = 0$. Then for any $v \in V$, consider the interior product $\iota_v(\alpha \wedge \omega^k)$. Note that we can *not* directly conclude that $\alpha = 0$ from $\alpha \wedge \omega^k = 0$ since $\alpha \in \bigwedge^{n-k+1} V^*$, not in $\bigwedge^{n-k} V^*$ as in the inductive hypothesis!)

Problem 10. Complete the following three problems on integrability.
(a) [4 points] State the Frobenius integrability theorem, in both the version based on vector fields and the version based on differential forms.
(b) [3 points] Consider $\mathbb{R}^3$ in the cylindrical coordinate $(r, \theta, z) \in \mathbb{R}_{\geq 0} \times [0, 2\pi) \times \mathbb{R}$. Prove that the following 2-dimensional distribution, defined pointwise as follows,

$$\mathcal{D}^2(r, \theta, z) := \mathrm{span}_{\mathbb{R}} \left\langle \cos r \frac{\partial}{\partial \theta} - r \sin r \frac{\partial}{\partial z}, \frac{\partial}{\partial r} \right\rangle$$

is *not* integrable anywhere (i.e., completely non-integrable).
(c) [3 points] Consider $\mathbb{R}^3$ in the standard coordinate (x, y, z). Prove that the following 2-dimensional distribution, defined pointwise as follows,

$$\mathcal{D}^2(x, y, z) = \mathrm{span}_{\mathbb{R}} \left\langle x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1) \frac{\partial}{\partial z}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right\rangle$$

is integrable.